# Violations of multisetting quaternion and octonion Bell inequalities

S. Kiesewetter, P. D. Drummond, and M. D. Reid

*Centre for Quantum-Atom Optics, Swinburne University of Technology, Melbourne, Australia*

(Received 9 May 2015; published 16 September 2015)

We examine the $N$-partite quaternion and octonion dichotomic Bell inequalities derived by Vogel and Shchukin utilizing the square identities of Euler and Degen, which apply to experiments with $M$ settings at each spatially separated site. We reveal these inequalities to be violated by the Greenberger-Horne-Zeilinger state, for $N \geqslant 3$ and $2 \leqslant M \leqslant 8$. Violations are robust with respect to loss, the threshold detection efficiency being $\eta > 2^{\frac{2}{N}-1}$ for all $M$, implying violations for efficiencies as low as $\eta \sim 50\%$ at each site, as $N \to \infty$.

DOI: 10.1103/PhysRevA.92.032118 PACS number(s): 03.65.Ud, 03.67.−a

## I. INTRODUCTION

Bell derived his famous inequalities as a way to distinguish the predictions of quantum mechanics from those of local hidden variable (LHV) theories [1,2]. Bell's original work examined two spatially separated spin-1/2 (qubit) systems. Experiments have supported quantum mechanics, which predicts a violation of the Bell inequality, thus falsifying all LHV theories [3–5]. Work on Bell nonlocality for multipartite systems (sites) began with the analyses of Svetlichny [6] and Greenberger, Horne, and Zeilinger (GHZ) [7]. Mermin showed that the degree of violation increased exponentially with the number of sites, $N$ [8]. This result translated to a surprising robustness to loss and noise for larger $N$ [9]. Experiments have realized $N$-partite GHZ states for $N \sim 6$–8 [10,11] and investigated their nonlocal properties for $N \sim 3$ [12].

Studies of Bell nonlocality to date have been mainly restricted to experiments with two measurement settings at each site [2,13,14]. Relatively little is known about Bell nonlocality for experiments with *multiple measurement settings*. Yet, multisetting experiments, first studied by Gisin and Collins [15,16], could have an important application to quantum information tasks. For example, quantum key distribution protocols depend on implementation of random independent switching between measurement settings at the relevant sites. Random switching between more measurement settings could increase security in some cases, by making it more difficult for Eve to gain correct knowledge of measurement choices. Also important is that for some applications such as device-independent cryptography, one may require rigorous violation of the relevant Bell inequality [17]. This is also essential for the rigorous falsification of all LHV theories [18,19]. Work by Brunner *et al.* [20] and Pal *et al.* [21] reveal multisetting Bell inequalities to be promising for this purpose.

Recent experiments have violated a Bell inequality without detection of fair sampling assumptions, using two measurement settings and detectors with efficiencies of $\eta \sim 75\%$ [19]. This violation was obtained using a nonmaximally entangled state [22,23]. Examination of LHV models gives a lower bound on the efficiencies $\eta$ required for violations of $M$-setting Bell inequalities as (at least) $\eta \geqslant 1/M$ [24–27], suggesting it may be possible to rigorously violate a Bell inequality at low efficiencies by using more than two settings. In fact, three-setting inequalities have been proposed for the two-site case, to allow a loophole-free violation with 43% at one detector (for the nonmaximally entangled state) [20,28]. Multisetting inequalities have been used for the loophole-free detection of the nonlocality associated with the Einstein-Podolsky-Rosen (EPR) paradox ("EPR steering" [29]), for efficiencies well below 50% [30,31] and consistent with the bound $\eta > 1/M$ [27]. For the maximally entangled state, LHV models exist to imply 50% as a feasible efficiency for symmetric Bell experiments [32,33], but it was initially unclear how one could violate a Bell inequality at this efficiency [20]. Subsequently, Cavalcanti *et al.* presented such an inequality for two settings that allowed violations at 50% for $N$-partite GHZ states as $N \to \infty$ [26,34]. In 2012, Pal *et al.* confirmed the advantage of multisettings, by deriving a family of multisetting Bell inequalities for $N$-partite GHZ states that predict threshold efficiencies as low as 38% for accessible parameters [21].

In this paper, we present a second set of multisetting multipartite Bell inequalities that are predicted to be violated for reasonable efficiencies. Shchukin and Vogel (SV) have derived two multisetting Bell inequalities, whose structure is closely related to the algebra of quaternions and octonions [35]. The two-setting version of these inequalities was originally derived by Cavalcanti-Foster-Reid-Drummond (CFRD) [36] and studied in the qubit form by Salles *et al.* [37] and Cavalcanti *et al.* [34]. For two settings, violations for $N$-partite qubit GHZ states are possible provided $N > 3$. These violations are different from those predicted by Mermin-Ardehali-Belinskii-Klyshko (MABK) [8,38,39], in the sense that they allow Bell nonlocality for reduced detection efficiencies, approaching the limit $\eta \to 50\%$ as $N \to \infty$ [26,34]. Salles *et al.* reported three-setting violations of generalizations of these inequalities, achieved using the GHZ state [37]. Up to now, however, as far as we know, there has been no explicit demonstration of violations of the quaternion or octonion Shchukin-Vogel Bell inequalities for higher settings.

Here, we show violation of the Shchukin-Vogel quaternion and octonion Bell inequalities using the GHZ state, for all $N \geqslant 3$ and for experiments involving $M = 2$–8 measurement settings. We show that the violations are predicted for efficiencies $\eta > 2^{\frac{2}{N}-1}$, independent of the number of measurement settings $M$. Although this does not give the optimistically expected dependence on $M$ [24,25], the efficiency approaches 50% for large number of sites for all $M$ and is obtained for the maximally entangled GHZ state based on inequalities similar to those of MABK. The case of the generalized GHZ state has been examined for multisetting generalizations of the MABK inequalities, where it was found that more than two settings can provide an advantage in detecting nonlocality [40–42]. We discuss this case for the SV inequalities, as well as the potential for experimental implementation, noting that heralding would be required.

## II. SHCHUKIN-VOGEL MULTISETTING BELL INEQUALITIES

We consider $N$ spatially separated systems and spacelike separated measurements performed on them. We suppose there are two measurement settings at each site labeled by $k$, so that observables $\hat{A}_k$ and $\hat{B}_k$ are measured by each setting. Then if local hidden variable (LHV) theories are valid, CFRD showed that the following inequality will hold [36]:

$$\left|\left\langle\prod_{k=1}^{N}(A_k+iB_k)\right\rangle\right|^2 \leqslant \left\langle\prod_{k=1}^{N}\{(A_k)^2+(B_k)^2\}\right\rangle, \tag{1}$$

where $A_k$ and $B_k$ are the results of the measurements $\hat{A}_k$ and $\hat{B}_k$. The moments that are required to be measured on the left side of the inequality are determined from the expansion involving the complex numbers $f_k = A_k + iB_k$. The moments for an arbitrary number of sites may be written using a recursive relation. Inequalities of this type have been studied further in Refs. [26,34,43–46] and adapted to give inequalities for other forms of nonlocality in Refs. [34,46,47]. Mermin originally formulated inequalities using the imaginary construction [8], and the left side of this inequality is closely related to that of the MABK inequalities [38,39]. For two sites, the CFRD inequality is

$$\begin{aligned}&\langle A_1A_2 - B_1B_2\rangle^2 + \langle B_1A_2 + A_1B_2\rangle^2\\ &\quad\leqslant \big\langle\big(A_1^2+B_1^2\big)\times\big(A_2^2+B_2^2\big)\big\rangle.\end{aligned} \tag{2}$$

Schukin and Vogel (SV) developed the theory further to derive a four and eight observable form of these inequalities [35]. We adopt their notation, denoting the outcomes of four observables that can be measured at each site $k$ by $A_k$, $B_k$, $C_k$, and $D_k$, and the outcomes of eight observables by $A_k$, $B_k$, $C_k$, $D_k$, $E_k$, $F_k$, $G_k$, and $H_k$. The quaternion multipartite inequalities are evaluated by forming the quaternion $q_k = A_k + iB_k + jC_k + kD_k$ on the left side and using quaternion multiplication. The octonion multipartite inequalities are similarly developed using the multiplication laws for octonions.

### Quaternion and octonion Bell inequalities

For two sites, the quaternionic Bell inequality is [35]

$$\begin{aligned}&\langle A_1A_2 - B_1B_2 - C_1C_2 - D_1D_2\rangle^2\\ &\quad+\langle B_1A_2 + A_1B_2 - D_1C_2 + C_1D_2\rangle^2\\ &\quad+\langle C_1A_2 - B_1D_2 + A_1C_2 + D_1B_2\rangle^2\\ &\quad+\langle D_1A_2 + A_1D_2 - C_1B_2 + B_1C_2\rangle^2\\ &\qquad\leqslant \big\langle\big(A_1^2+B_1^2+C_1^2+D_1^2\big)\big(A_2^2+B_2^2+C_2^2+D_2^2\big)\big\rangle.\end{aligned} \tag{3}$$

For the two-side case, we are considering the Bell inequality

$$\begin{aligned}&|\langle(\hat{A}_1+i\hat{B}_1+j\hat{C}_1+k\hat{D}_1)(\hat{A}_2+i\hat{B}_2+j\hat{C}_2+k\hat{D}_2)\rangle|^2\\ &\quad\leqslant \big\langle\big(\hat{A}_1^2+\hat{B}_1^2+\hat{C}_1^2+\hat{D}_1^2\big)\big(\hat{A}_2^2+\hat{B}_2^2+\hat{C}_2^2+\hat{D}_2^2\big)\big\rangle,\end{aligned} \tag{4}$$

where here the well-known quaternion algebra $i^2 = j^2 = k^2 = ijk = -1$, $ij = -ji = k, jk = -kj = i$ applies. This inequality holds based on the fact that for any observable $\hat{A}$, $\langle\hat{A}\rangle^2 \leqslant \langle\hat{A}^2\rangle$. Recursive application of (4) allows for arbitrarily many ($N$) sites. The product $(\hat{A}_1 + i\hat{B}_1 + j\hat{C}_1 + k\hat{D}_1)(\hat{A}_2 + i\hat{B}_2 + j\hat{C}_2 + k\hat{D}_2)$ can be expanded into

$$\begin{aligned}&(\hat{A}_1+i\hat{B}_1+j\hat{C}_1+k\hat{D}_1)(\hat{A}_2+i\hat{B}_2+j\hat{C}_2+k\hat{D}_2)\\ &\quad= \hat{I}_r + i\cdot\hat{I}_i + j\cdot\hat{I}_j + k\cdot\hat{I}_k,\end{aligned} \tag{5}$$

where

$$\begin{aligned}\hat{I}_r &= \hat{A}_1\hat{A}_2 - \hat{B}_1\hat{B}_2 - \hat{C}_1\hat{C}_2 - \hat{D}_1\hat{D}_2,\\ \hat{I}_i &= \hat{A}_1\hat{B}_2 + \hat{B}_1\hat{A}_2 + \hat{C}_1\hat{D}_2 - \hat{D}_1\hat{C}_2,\\ \hat{I}_j &= \hat{A}_1\hat{C}_2 - \hat{B}_1\hat{D}_2 + \hat{C}_1\hat{A}_2 + \hat{D}_1\hat{B}_2,\\ \hat{I}_k &= \hat{A}_1\hat{D}_2 + \hat{B}_1\hat{C}_2 - \hat{C}_1\hat{B}_2 + \hat{D}_1\hat{A}_2.\end{aligned} \tag{6}$$

The terms $I_r, I_i, I_j, I_k$ satisfy the Euler square identity

$$\begin{aligned}&\hat{I}_r^2 + \hat{I}_i^2 + \hat{I}_j^2 + \hat{I}_k^2\\ &\quad= \big(\hat{A}_1^2+\hat{B}_1^2+\hat{C}_1^2+\hat{D}_1^2\big)\big(\hat{A}_2^2+\hat{B}_2^2+\hat{C}_2^2+\hat{D}_2^2\big),\end{aligned} \tag{7}$$

which is fundamental for inequality (4) to hold [35].

The three-site quaternion Bell inequality is derived similarly, and its expanded form is written explicitly as [35]

$$\begin{aligned}&(I_r)^2 + (I_i)^2 + (I_j)^2 + (I_k)^2\\ &\quad\leqslant \big\langle\big(A_1^2+B_1^2+C_1^2+D_1^2\big)\big(A_2^2+B_2^2+C_2^2+D_2^2\big)\\ &\qquad\times\big(A_3^2+B_3^2+C_3^2+D_3^2\big)\big\rangle,\end{aligned} \tag{8}$$

where

$$\begin{aligned}I_r &= \langle(A_1A_2 - B_1B_2 - C_1C_2 - D_1D_2)A_3\\ &\quad-(B_1A_2 + A_1B_2 - D_1C_2 + C_1D_2)B_3\\ &\quad-(C_1A_2 - B_1D_2 + A_1C_2 + D_1B_2)C_3\\ &\quad-(D_1A_2 + A_1D_2 - C_1B_2 + B_1C_2)D_3\rangle,\\[4pt] I_i &= \langle(A_1A_2 - B_1B_2 - C_1C_2 - D_1D_2)B_3\\ &\quad+(B_1A_2 + A_1B_2 - D_1C_2 + C_1D_2)A_3\\ &\quad+(C_1A_2 - B_1D_2 + A_1C_2 + D_1B_2)D_3\\ &\quad-(D_1A_2 + A_1D_2 - C_1B_2 + B_1C_2)C_3\rangle,\\[4pt] I_j &= \langle(A_1A_2 - B_1B_2 - C_1C_2 - D_1D_2)C_3\\ &\quad-(B_1A_2 + A_1B_2 - D_1C_2 + C_1D_2)D_3\\ &\quad+(C_1A_2 - B_1D_2 + A_1C_2 + D_1B_2)A_3\\ &\quad+(D_1A_2 + A_1D_2 - C_1B_2 + B_1C_2)B_3\rangle,\\[4pt] I_k &= \langle(A_1A_2 - B_1B_2 - C_1C_2 - D_1D_2)D_3\\ &\quad+(B_1A_2 + A_1B_2 - D_1C_2 + C_1D_2)C_3\\ &\quad-(C_1A_2 - B_1D_2 + A_1C_2 + D_1B_2)B_3\\ &\quad+(D_1A_2 + A_1D_2 - C_1B_2 + B_1C_2)A_3\rangle.\end{aligned} \tag{9}$$

The $N$-site inequality is readily evaluated in a recursive form that enables computation.

We define the quantity $B_{N,M}$ as the ratio of the left-hand side (lhs) to the right-hand side (rhs) of the Bell inequality

[Eq. (1), Eq. (3), or Eq. (8), etc.] that has $M$ settings and $N$ sites, so that

$$B_{N,M} > 1 \tag{10}$$

indicates a violation of the Bell inequality. The expressions for the octonion Bell inequalities are derived similarly, using the eight square identity as explained by Shchukin and Vogel [35]. A setup with three, five, six, and seven settings can be achieved by simply defining the remaining settings as identical to zero. For example, a setup of three settings can be achieved by setting all $D_i = 0$.

## III. QUANTUM PREDICTIONS FOR GHZ STATES

We suppose the multipartite system is $N$ spin-1/2 subsystems, prepared in a GHZ state. The GHZ state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\{|\uparrow\rangle^{\otimes N} \pm |\downarrow\rangle^{\otimes N}\} \tag{11}$$

is a maximally entangled state for $N$ spin-1/2 systems. The Pauli spin observables applying to a measurement on the system at the $k$th site are denoted $\hat{\sigma}_x{}^k$, $\hat{\sigma}_y{}^k$, $\hat{\sigma}_z{}^k$. Here, $|\uparrow\rangle_k$ and $|\downarrow\rangle_k$ denote the eigenstates of $\hat{\sigma}_z^k$. We suppose the possible measurements are the set of all possible spin observables. The GHZ superposition state has been generally defined here with either sign in the superposition of probability amplitudes. However, as explained later, the final results for the Bell violations do not depend on the sign.

The right side of the inequalities (1) and (4) simplify, for the Pauli spin experiment where the results of the measurements are always $\pm 1$. For the case of two settings (observables), the right side of the inequality reduces to $2^N$. For four settings, the right side reduces to $4^N$, and for eight settings, it reduces to $8^N$.

To evaluate the left side of the inequalities, we need to calculate the moments of the GHZ state. For example, for $N = 2$, we substitute in (4) the operators $\hat{A}_k = \hat{n}_{1k} \cdot \hat{\sigma}^k$, $\hat{B}_k = \hat{n}_{2k} \cdot \hat{\sigma}^k$, $\hat{C}_k = \hat{n}_{3k} \cdot \hat{\sigma}^k$, and $\hat{D}_k = \hat{n}_{4k} \cdot \hat{\sigma}^k$, where $\hat{\sigma}^k$ is understood to act only on particle $k$, and $\hat{n}_{ij}$ are arbitrary unit-length vectors. One can rewrite $\hat{n}_\alpha \cdot \hat{\sigma}$ as $\sin(\theta_\alpha)\cos(\varphi_\alpha)\hat{\sigma}_x + \sin(\theta_\alpha)\sin(\varphi_\alpha)\hat{\sigma}_y + \cos(\theta_\alpha)\hat{\sigma}_z$. Considering a typical term $\langle\Psi^\pm|\hat{\sigma}_i^1\hat{\sigma}_j^2|\Psi^\pm\rangle$ where $i,j = \{x,y,z\}$ and $|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\uparrow\rangle_2 \pm |\downarrow\rangle_1|\downarrow\rangle_2)$, as they appear in the left side of (4), one finds that the only nonzero terms are $\langle\hat{\sigma}_x^1\hat{\sigma}_x^2\rangle = \pm 1$, $\langle\hat{\sigma}_y^1\hat{\sigma}_y^2\rangle = \mp 1$, and $\langle\hat{\sigma}_z^1\hat{\sigma}_z^2\rangle = 1$. Hence, a term $\langle\hat{X}_{i1}\hat{X}_{j2}\rangle$ will reduce to

$$\begin{aligned}\langle\hat{X}_{i1}\hat{X}_{j2}\rangle = {} & \sin(\theta_{i1})\sin(\theta_{j2}) \\ & \times[\pm\cos(\varphi_{i1})\cos(\varphi_{j2}) \mp \sin(\varphi_{i1})\sin(\varphi_{j2})] \\ & + \cos(\theta_{i1})\cos(\theta_{j2}),\end{aligned} \tag{12}$$

where $\hat{X}_1 = \hat{A}$, $\hat{X}_2 = \hat{B}$, $\hat{X}_3 = \hat{C}$, and $\hat{X}_4 = \hat{D}$. Similarly, for evaluating the predictions at larger $N$, only some correlations will be nonzero. The nonzero terms are as given in Table I. Note that because of the alternating nature of the terms in Table I, for a system with all polar angles fixed at $\theta = \frac{\pi}{4}$, the relative sign in the GHZ state produces an overall sign that affects all terms in Table I equally. This means that for such a setup, the left side of the SV inequality will be independent of the sign of the GHZ state due to the squaring of its terms.

TABLE I. Evaluation of Pauli products, showing the nonzero moments predicted by the GHZ state for various $N$. Here, every combination of $\hat{\sigma}_{xyz}$ acting on different particles must be considered as nonzero. For example, the term $\langle\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_y\rangle$ means that $\langle\hat{\sigma}_x^1\hat{\sigma}_y^2\hat{\sigma}_y^3\rangle$, $\langle\hat{\sigma}_y^1\hat{\sigma}_x^2\hat{\sigma}_y^3\rangle$, and $\langle\hat{\sigma}_y^1\hat{\sigma}_y^2\hat{\sigma}_x^3\rangle$ must be taken into consideration.

| $N$ | |
|---|---|
| 3 | $\langle\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\rangle = \pm 1$ |
| | $\langle\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_y\rangle = \mp 1$ |
| 4 | $\langle\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\rangle = \pm 1$ |
| | $\langle\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_y\rangle = \mp 1$ |
| | $\langle\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y\rangle = \pm 1$ |
| | $\langle\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_z\rangle = 1$ |
| 5 | $\langle\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\rangle = \pm 1$ |
| | $\langle\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_y\rangle = \mp 1$ |
| | $\langle\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y\rangle = \pm 1$ |
| 6 | $\langle\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\rangle = \pm 1$ |
| | $\langle\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_y\rangle = \mp 1$ |
| | $\langle\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y\rangle = \pm 1$ |
| | $\langle\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y\rangle = \mp 1$ |
| | $\langle\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_z\hat{\sigma}_z\rangle = 1$ |
| 7 | $\langle\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\rangle = \pm 1$ |
| | $\langle\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_y\rangle = \mp 1$ |
| | $\langle\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y\rangle = \pm 1$ |
| | $\langle\hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y\hat{\sigma}_y\rangle = \mp 1$ |

When $N = 2$, the GHZ state reduces to a Bell state. In this case, no violation of the two-site inequality is possible, consistent with results in Ref. [37]. As an example, we could take $A = \hat{\sigma}_x$ and $B = \hat{\sigma}_y$, $C_1 = \frac{1}{\sqrt{2}}\{-\hat{\sigma}_x + \hat{\sigma}_y\}$, $D_1 = \frac{1}{\sqrt{2}}\{-\hat{\sigma}_x - \hat{\sigma}_y\}$, and $C_2 = \frac{1}{\sqrt{2}}\{\hat{\sigma}_x + \hat{\sigma}_y\}$, $D_2 = \frac{1}{\sqrt{2}}\{\hat{\sigma}_x - \hat{\sigma}_y\}$. Then we find $C_1C_2 - D_1D_2 = A_1A_2 - B_1B_2 = \hat{\sigma}_x^1\hat{\sigma}_x^2 - \hat{\sigma}_y^1\hat{\sigma}_y^2$, which gives the maximum of 4, and $-D_1C_2 + C_1D_2 = B_1A_2 + A_1B_2 = \hat{\sigma}_x^1\hat{\sigma}_y^2 + \hat{\sigma}_y^1\hat{\sigma}_x^2$ (which will give 0), and $C_1A_2 - B_1D_2 + A_1C_2 + D_1B_2 = 0$ and $D_1A_2 + A_1D_2 - C_1B_2 + B_1C_2$, so the left and right sides of the inequality are equal. That there is no violation is confirmed using the full search over all possible spin measurement choices.

Violations of the SV Bell inequality become possible for $N \geqslant 3$. We can verify algebraically that the choice of measurements of $C_1 = C_2 = \frac{1}{2}\{\hat{\sigma}_x + \sqrt{3}\hat{\sigma}_y\}$, $D_1 = -D_2 = \frac{1}{2}\{\sqrt{3}\hat{\sigma}_x - \hat{\sigma}_y\}$, $C_3 = \frac{1}{2}\{-\hat{\sigma}_x + \sqrt{3}\hat{\sigma}_y\}$, and $D_3 = \frac{1}{2}\{\sqrt{3}\hat{\sigma}_x + \hat{\sigma}_y\}$ will give a $-4$ for both of the first two terms in the $I_r$ . However, the last two terms will be identically zero for this choice of measurement, so $I_r = -8$. We see $I_i = 0$, $I_j = -4$, and $I_k = -4\sqrt{3}$. The result is that the left side of the inequality is $2 \times 8^2$ and the right side is $4^3$, which gives a significant violation. The numerical search over all possible angles confirmed the result as that giving the maximum violation.

For $N$ sites and $M$ settings, the right side of the SV Bell inequality becomes rhs $= M^N$. Our numerical analysis has shown to great accuracy for $N = 2,3,\ldots,7$ sites and $M =$

TABLE II. Optimal measurement settings $\phi_{kn}$, $k = A,B,C,D$, at each site $n = 1,\ldots,N$. The settings are given in spherical coordinates, with all $\theta \equiv 0$ at all sites, so that, e.g., $\hat{A}_1 = (\cos\phi_{A1}, \sin\phi_{A1}, 0)^T \times \hat{\sigma}^1 = \cos\phi_{A1}\hat{\sigma}_x^1 + \sin\phi_{A1}\hat{\sigma}_y^1$. A higher number of sites $N$ can be treated by simply continuing the alternating pattern. The settings for higher $M$ are given in Tables III–V.

| $N$ | | 2 settings $\phi$ | | | 3 settings $\phi$ | | |
|---|---|---|---|---|---|---|---|
| | | $A$ | $B$ | | $A$ | $B$ | $C$ |
| 3 | 1 | 0 | $\frac{\pi}{2}$ | 1 | 0 | $\frac{2\pi}{3}$ | $\frac{\pi}{3}$ |
| | 2 | 0 | $\frac{\pi}{2}$ | 2 | 0 | $\frac{\pi}{3}$ | $\frac{2\pi}{3}$ |
| | 3 | 0 | $\frac{\pi}{2}$ | 3 | 0 | $\frac{2\pi}{3}$ | $\frac{\pi}{3}$ |
| | | $A$ | $B$ | | $A$ | $B$ | $C$ |
| 4 | 1 | 0 | $\frac{\pi}{2}$ | 1 | 0 | $\frac{2\pi}{3}$ | $\frac{\pi}{3}$ |
| | 2 | 0 | $\frac{\pi}{2}$ | 2 | 0 | $\frac{\pi}{3}$ | $\frac{2\pi}{3}$ |
| | 3 | 0 | $\frac{\pi}{2}$ | 3 | 0 | $\frac{2\pi}{3}$ | $\frac{\pi}{3}$ |
| | 4 | 0 | $\frac{\pi}{2}$ | 4 | 0 | $\frac{\pi}{3}$ | $\frac{2\pi}{3}$ |

$2,3,\ldots,8$ settings that the angle choices stated in Tables II–V yield a maximal value for the left-hand side of the SV Bell inequalities, which is given by lhs $= M^N \times 2^{N-2}$. The rhs of the inequality is rhs $= M^N$. Not taking detector inefficiencies into account, we get a Bell violation value of $\frac{\text{lhs}}{\text{rhs}} = 2^{N-2}$, as plotted in Fig. 1, regardless of the number of settings used. We note that while the SV Bell inequalities are derived based on four and eight observables, using quaternion and octonion algebra, respectively, we can by setting one or more of the observables to zero in each of them arrive at Bell inequalities for fewer settings. In this way, the quaternion Bell inequality is also a Bell inequality for $M = 2,3,4$ settings, reducing in the case of $M = 2$ to the CFRD Bell inequality for spins [34,37]. Similarly, the octonion SV Bell inequality gives rise to the Bell inequalities with $M = 5,6,7$, and 8 settings.

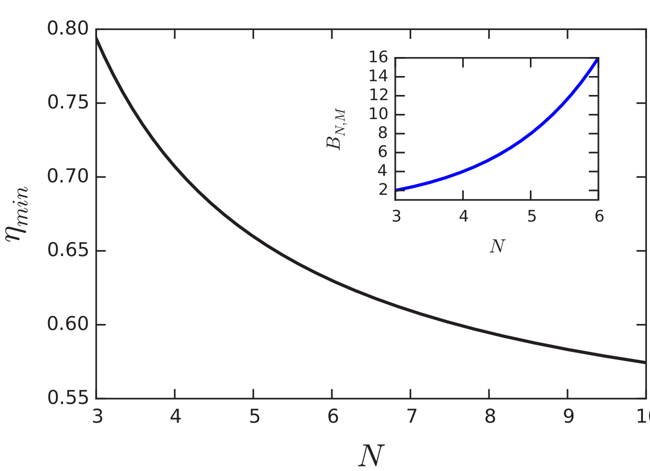


FIG. 1. (Color online) Threshold detector efficiency for $N$ sites: The minimal detector efficiency $\eta_{\min}$ needed to observe Bell violations for a given number of sites $N$. Inset: The Bell value $B_{N,M}$ vs $N$. $B_{N,M} > 1$ indicates violation of the Bell inequality. The results are independent of the number of measurement settings $M$.

## IV. DETECTION INEFFICIENCIES, NOISE AND HERALDING

We now examine the important question of the effect of detection inefficiencies on the Bell violations. The following analysis will model the loss that takes place due to detector inefficiencies and propagation through the medium. We extend the analysis given in Ref. [34]. We represent the experimental observables as the Schwinger spin observables,

$$\begin{aligned}
\hat{\sigma}_z &= \hat{a}_+^\dagger\hat{a}_+ - \hat{a}_-^\dagger\hat{a}_-,\\
\hat{\sigma}_x &= \hat{a}_+^\dagger\hat{a}_- + \hat{a}_+\hat{a}_-^\dagger,\\
\hat{\sigma}_y &= (\hat{a}_+^\dagger\hat{a}_- - \hat{a}_+\hat{a}_-^\dagger)/i,\\
\hat{\sigma}^2 &= \hat{n}(\hat{n}+2),\\
\hat{n} &= \hat{a}_+^\dagger\hat{a}_+ + \hat{a}_-^\dagger\hat{a}_-,
\end{aligned} \tag{13}$$

where $\hat{a}_\pm$ is the destruction boson operator for two orthogonal modes $\pm$ at the site $k$. Here, $\hat{\sigma}^2 = (\hat{\sigma}_x)^2 + (\hat{\sigma}_y)^2 + (\hat{\sigma}_z)^2$, and $\hat{n}$ is the total number operator for each site. The superscripts $k$ denote which site is being referred to. Usually, the modes $\pm$ correspond to orthogonal field polarization and $\hat{a}_\pm^{k\dagger}\hat{a}_\pm^k$ is the number operator for a photonic mode $\pm$ at site $k$. At each site, the spin measurement $\hat{\sigma}_\phi^k$ is performed by a polarization measurement chosen at a suitable angle (and possibly with phase shifts) [48]. For example, the modes at the detectors after the polarizer measurement for $\hat{\sigma}_\phi$ can be represented by the transformed modes $\hat{c}_+$, $\hat{c}_-$ which are linear combinations of $\hat{a}_\pm$ and for which $\hat{c}_+^\dagger\hat{c}_+ - \hat{c}_-^\dagger\hat{c}_- = \hat{\sigma}_\phi$. The photonic GHZ state is

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle^{\otimes N} + e^{i\phi}|0\rangle^{\otimes N}), \tag{14}$$

where $|n\rangle^{\otimes N} \equiv \bigotimes_{k=1}^N |n\rangle_k$ and $|n\rangle_k$ are the eigenstates of $\hat{n}^k$. We see that components of the photonic GHZ state are mapped into the $\pm1$ eigenstates of $\hat{\sigma}_z$: $|\downarrow\rangle_k \rightarrow |0\rangle_{+k}|1\rangle_{-k}$ and $|\uparrow\rangle_k \rightarrow |1\rangle_{+k}|0\rangle_{-k}$.

With nonideal detection efficiencies, not all photons that impinge on the detector will be recorded. For each emission event, there are three outcomes at each site. We follow the traditional example [2] and denote the outcomes of the observables $A$, $B$, etc. as $+1$, $-1$, or 0 if the photon is detected "up," "down," or "not at all," respectively. There are two detectors at each site, placed to determine the number of photons (0 or 1) corresponding to modes $\hat{c}_+^{(k)}$ and $\hat{c}_-^{(k)}$, respectively (see Refs. [2,49] for details). We model the inefficiency (loss) of each detector using a simple beam-splitter model, where we evaluate the moments of detected fields $\hat{a}_{\text{det}}$ given by

$$\hat{a}_{\text{det}} = \sqrt{\eta}\hat{a} + \sqrt{1-\eta}\hat{a}_{\text{vac}}. \tag{15}$$

Here, $\eta$ is the probability that an incoming photon is detected. The $\hat{a}$ is the boson operator for the incoming field mode, and $\hat{a}_{\text{vac}}$ is the boson operator for a vacuum reservoir mode that couples to the incoming field and into which quanta are lost.

Equation (15) enables calculation of the predicted effect of detection inefficiencies on the moments needed for the Bell inequalities. It is important to understand how the experiment would be carried out. In a LHV theory, the quantities $A, B, \ldots$

TABLE III. Optimal angles for four and five measurement settings, as described for Table II.

| $N$ | | 4 settings $\phi$ | | | | | 5 settings $\phi$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $A$ | $B$ | $C$ | $D$ | | $A$ | $B$ | $C$ | $D$ | $E$ |
| 3 | 1 | 0 | $\frac{\pi}{6}$ | $-\frac{\pi}{3}$ | $\frac{\pi}{2}$ | 1 | 0 | $\frac{\pi}{6}$ | $-\frac{\pi}{3}$ | $\frac{4\pi}{3}$ | $\frac{2\pi}{3}$ |
| | 2 | 0 | $\frac{5\pi}{6}$ | $\frac{4\pi}{3}$ | $\frac{\pi}{2}$ | 2 | 0 | $\frac{5\pi}{6}$ | $\frac{4\pi}{3}$ | $-\frac{\pi}{3}$ | $\frac{\pi}{3}$ |
| | 3 | 0 | $\frac{\pi}{6}$ | $-\frac{\pi}{3}$ | $\frac{\pi}{2}$ | 3 | 0 | $\frac{\pi}{6}$ | $-\frac{\pi}{3}$ | $\frac{4\pi}{3}$ | $\frac{2\pi}{3}$ |
| | | $A$ | $B$ | $C$ | $D$ | | $A$ | $B$ | $C$ | $D$ | $E$ |
| 4 | 1 | 0 | $\frac{\pi}{6}$ | $-\frac{\pi}{3}$ | $\frac{\pi}{2}$ | 1 | 0 | $\frac{\pi}{6}$ | $-\frac{\pi}{3}$ | $\frac{4\pi}{3}$ | $\frac{2\pi}{3}$ |
| | 2 | 0 | $\frac{5\pi}{6}$ | $\frac{4\pi}{3}$ | $\frac{\pi}{2}$ | 2 | 0 | $\frac{5\pi}{6}$ | $\frac{4\pi}{3}$ | $-\frac{\pi}{3}$ | $\frac{\pi}{3}$ |
| | 3 | 0 | $\frac{\pi}{6}$ | $-\frac{\pi}{3}$ | $\frac{\pi}{2}$ | 3 | 0 | $\frac{\pi}{6}$ | $-\frac{\pi}{3}$ | $\frac{4\pi}{3}$ | $\frac{2\pi}{3}$ |
| | 4 | 0 | $\frac{5\pi}{6}$ | $\frac{4\pi}{3}$ | $\frac{\pi}{2}$ | 4 | 0 | $\frac{5\pi}{6}$ | $\frac{4\pi}{3}$ | $-\frac{\pi}{3}$ | $\frac{\pi}{3}$ |

that appear in the inequality are numbers that now have the possible values $\pm 1,0$. The derivation of the Bell SV inequalities does not restrict the values that the variables can attain, and hence the same inequalities will still hold rigorously in the lossy scenario, to allow a loophole-free test of LHV theories. In an ideal situation, one *heralds* the emission event and detects the value of the observables $A,B,\ldots$ at each of the sites, some being $+1$ or $-1$, and some being 0. One evaluates from the data the moments that are defined on the left and right sides of the inequality. If these moments violate the inequalities, then we have confirmed the failure of all LHV theories to describe the experiment.

What do we expect to obtain for a given detection inefficiency $\eta$? On the lhs of the inequalities, we measure moments like those in the Clauser-Horne-Shimony-Holt (CHSH) Bell inequalities [2,13], such as $\langle A_1A_2\cdots A_N\rangle$. Consider first two sites. We denote the joint probability for the outcome $+1$ and $-1$ at sites 1 and 2, respectively, by $P(+,-)$, etc. Thus,

$$\langle A_1A_2\rangle = P(+,+) + P(-,-) - P(+,-) - P(-,+). \quad (16)$$

The probabilities where the outcome is the null event 0 do not directly contribute because their value is $A = 0$, but the null events contribute to the total number of counts and hence to the normalization of the probabilities. The joint probabilities $P(+,+)$, etc. involve a single photon detection at each site, and hence will scale as $\eta^2$. More generally, with $N$ sites, there will be $N$ photon detections and we will obtain a scaling of $\eta^N$. Hence we obtain, for the left side of the Bell inequalities,

TABLE IV. Optimal angles for six and seven measurement settings, as described for Table II.

| $N$ | | 6 settings $\phi$ | | | | | | | 7 settings $\phi$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ |
| 3 | 1 | 0 | $\frac{\pi}{8}$ | $\frac{3\pi}{8}$ | $\frac{5\pi}{8}$ | $\frac{7\pi}{8}$ | $\frac{\pi}{2}$ | 1 | 0 | $\frac{\pi}{8}$ | $\frac{3\pi}{8}$ | $\frac{5\pi}{8}$ | $\frac{7\pi}{8}$ | $-\frac{\pi}{3}$ | $\frac{4\pi}{3}$ |
| | 2 | 0 | $\frac{7\pi}{8}$ | $\frac{5\pi}{3}$ | $\frac{3\pi}{8}$ | $\frac{\pi}{8}$ | $\frac{\pi}{2}$ | 2 | 0 | $\frac{7\pi}{8}$ | $\frac{5\pi}{8}$ | $\frac{3\pi}{8}$ | $\frac{\pi}{8}$ | $\frac{4\pi}{3}$ | $-\frac{\pi}{3}$ |
| | 3 | 0 | $\frac{\pi}{8}$ | $\frac{3\pi}{8}$ | $\frac{5\pi}{8}$ | $\frac{7\pi}{8}$ | $\frac{\pi}{2}$ | 3 | 0 | $\frac{\pi}{8}$ | $\frac{3\pi}{8}$ | $\frac{5\pi}{8}$ | $\frac{7\pi}{8}$ | $-\frac{\pi}{3}$ | $\frac{4\pi}{3}$ |
| | | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ |
| 4 | 1 | 0 | $\frac{\pi}{8}$ | $\frac{3\pi}{8}$ | $\frac{5\pi}{8}$ | $\frac{7\pi}{8}$ | $\frac{\pi}{2}$ | 1 | 0 | $\frac{\pi}{8}$ | $\frac{3\pi}{8}$ | $\frac{5\pi}{8}$ | $\frac{7\pi}{8}$ | $-\frac{\pi}{3}$ | $\frac{4\pi}{3}$ |
| | 2 | 0 | $\frac{7\pi}{8}$ | $\frac{5\pi}{3}$ | $\frac{3\pi}{8}$ | $\frac{\pi}{8}$ | $\frac{\pi}{2}$ | 2 | 0 | $\frac{7\pi}{8}$ | $\frac{5\pi}{8}$ | $\frac{3\pi}{8}$ | $\frac{\pi}{8}$ | $\frac{4\pi}{3}$ | $-\frac{\pi}{3}$ |
| | 3 | 0 | $\frac{\pi}{8}$ | $\frac{3\pi}{8}$ | $\frac{5\pi}{8}$ | $\frac{7\pi}{8}$ | $\frac{\pi}{2}$ | 3 | 0 | $\frac{\pi}{8}$ | $\frac{3\pi}{8}$ | $\frac{5\pi}{8}$ | $\frac{7\pi}{8}$ | $-\frac{\pi}{3}$ | $\frac{4\pi}{3}$ |
| | 4 | 0 | $\frac{7\pi}{8}$ | $\frac{5\pi}{3}$ | $\frac{3\pi}{8}$ | $\frac{\pi}{8}$ | $\frac{\pi}{2}$ | 4 | 0 | $\frac{7\pi}{8}$ | $\frac{5\pi}{8}$ | $\frac{3\pi}{8}$ | $\frac{\pi}{8}$ | $\frac{4\pi}{3}$ | $-\frac{\pi}{3}$ |

TABLE V. Optimal angles for eight measurement settings, as described for Table II.

| $N$ | | 8 settings $\phi$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ | $H$ |
| 3 | 1 | 0 | $\frac{\pi}{8}$ | $\frac{3\pi}{8}$ | $\frac{5\pi}{8}$ | $\frac{7\pi}{8}$ | $-\frac{\pi}{8}$ | $-\frac{5\pi}{8}$ | $\frac{3\pi}{2}$ |
| | 2 | 0 | $\frac{7\pi}{8}$ | $\frac{5\pi}{8}$ | $\frac{3\pi}{8}$ | $\frac{\pi}{8}$ | $\frac{9\pi}{8}$ | $\frac{13\pi}{8}$ | $-\frac{\pi}{2}$ |
| | 3 | 0 | $\frac{\pi}{8}$ | $\frac{3\pi}{8}$ | $\frac{5\pi}{8}$ | $\frac{7\pi}{8}$ | $-\frac{\pi}{8}$ | $-\frac{5\pi}{8}$ | $\frac{3\pi}{2}$ |
| | | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ | $H$ |
| 4 | 1 | 0 | $\frac{\pi}{8}$ | $\frac{3\pi}{8}$ | $\frac{5\pi}{8}$ | $\frac{7\pi}{8}$ | $-\frac{\pi}{8}$ | $-\frac{5\pi}{8}$ | $\frac{3\pi}{2}$ |
| | 2 | 0 | $\frac{7\pi}{8}$ | $\frac{5\pi}{8}$ | $\frac{3\pi}{8}$ | $\frac{\pi}{8}$ | $\frac{9\pi}{8}$ | $\frac{13\pi}{8}$ | $-\frac{\pi}{2}$ |
| | 3 | 0 | $\frac{\pi}{8}$ | $\frac{3\pi}{8}$ | $\frac{5\pi}{8}$ | $\frac{7\pi}{8}$ | $-\frac{\pi}{8}$ | $-\frac{5\pi}{8}$ | $\frac{3\pi}{2}$ |
| | 4 | 0 | $\frac{7\pi}{8}$ | $\frac{5\pi}{8}$ | $\frac{3\pi}{8}$ | $\frac{\pi}{8}$ | $\frac{9\pi}{8}$ | $\frac{13\pi}{8}$ | $-\frac{\pi}{2}$ |

lhs $= \eta^{2N}M^N \times 2^{N-2}$. On the other hand, the moments on the rhs involve the expectation values of the $A^2$ values. The $A^2$ value is always 1 if the photon is detected either "up" or "down," and is always 0 if the photon is not detected. Denoting the probabilities for the outcomes $+1$, $-1$, and 0 at site $k$ by $P_k(+)$, $P_k(-)$, and $P_k(0)$, respectively, we note that

$$\langle A_1^2A_2^2\rangle = P(+,+) + P(+,-) + P(-,+) + P(-,-). \quad (17)$$

The moment scales with efficiency as $\eta^2$. Extending to the product of results for $\langle A_1^2A_2^2\ldots\rangle$ based on measurement at the $N$ sites, we see the prediction is rhs $= \eta^N M^N$. This means that an experiment involving $N$ sites requires a minimum detector efficiency of $\eta_{\min} = 2^{\frac{2}{N}-1}$ in order to detect Bell violation, regardless of the number of settings $M$ used (Fig. 1).

We note that because of the square on the lhs of the SV Bell inequalities, the violation cannot straightforwardly be obtained without determining the probability of the zero detection event at all sites. Thus, in the experiment described, there is a need for heralding the emission events in order to obtain a violation of the inequalities [50]. This provides a substantial limitation. Alternative experimental arrangements could be explored in the future. For example, it is possible that an analysis of the type presented by Garg and Mermin for the CHSH inequalities may be useful in providing an alternative experiment, without the need for heralding [51].

As a further calculation of the sensitivity of the Bell violations to possible noise sources, we show in Fig. 2 the effect of Gaussian random noise on the measurement angle. Regardless of the number of settings, the violations are robust to this noise source.

### V. GENERALIZED GHZ STATES

Now we turn to the generalized GHZ state

$$|\Psi\rangle = \cos\alpha|\uparrow\rangle^{\otimes N} + \sin\alpha|\downarrow\rangle^{\otimes N}. \quad (18)$$

It has been shown for the bipartite ($N = 2$) case that the nonmaximally entangled state (which has $\alpha < \pi/4$) allows violations of Bell inequalities for *lower* efficiency thresholds than the maximally entangled state ($\alpha = \pi/4$) [23]. These violations are achieved with asymmetric Clauser-Horne-type inequalities [19,20,22,28]. Also interesting is that the MABK Bell inequalities do not allow any violation for the generalized GHZ state $|\Phi\rangle = \cos\alpha|000\rangle + \sin\alpha|111\rangle$ for some values of

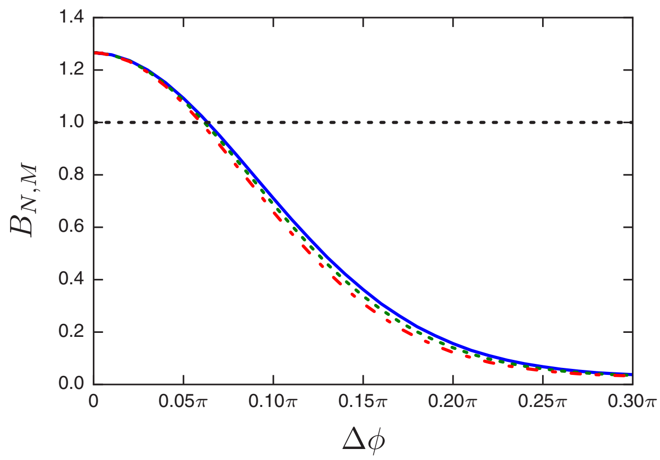


FIG. 2. (Color online) Effect of measurement noise: The Bell value $B_{N,M}$ achievable for $N = 4$ sites and (from top to lower line) $M = 2,3,8$ measurement settings when the ensemble averages are evaluated with a random Gaussian fluctuation of standard deviation $\Delta\phi$ in the measurement angles. Here, detector efficiency is $\eta = 0.75$. Violation of the Bell inequality is achieved when $B_{N,M} > 1$. The mean angle setting is optimized according to the values given in Tables II and V, respectively.

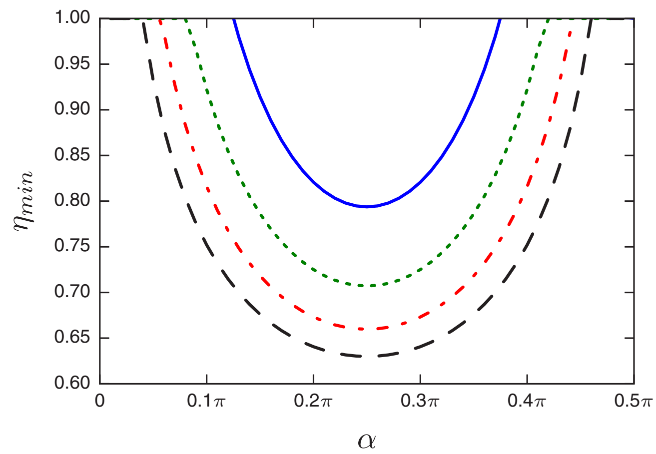


FIG. 4. (Color online) Threshold detector efficiencies for the generalized GHZ state: Here, $B_{N,M} > 1$ indicates violation of the Bell inequality with (from top to lower line) $N = 3,4,5,6$ sites and $M = 2$ measurement settings. The results for settings $M = 2$–8 are indistinguishable.

$\alpha$, namely $\sin 2\alpha \leqslant 1/\sqrt{2^{N-1}}$ and $N$ odd [40]. However, a violation becomes possible on using extensions of the MABK inequalities for four measurement settings [41,42].

We are thus motivated to ask the following question: For what range of $\alpha$ does the two-setting CFRD-type Bell inequality allow a test of nonlocality, and can this range be extended (as in the MABK case) by considering multiple settings? Our answer to this question is shown in Figs. (3) and (4).

For detection efficiencies at $\eta \sim 0.65$, one would require GHZ states with $N \geqslant 6$ to obtain a violation of the Bell inequalities. This would not seem unrealistic, given the experiments of Refs. [10] which report creation of photonic GHZ states for $N = 6$–8. A challenging feature of the experimental implementation would also be to correctly measure the probability of the null (zero-detection) event.

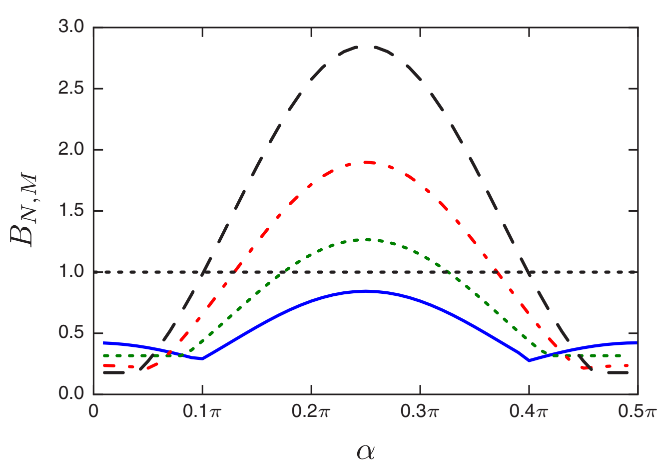


FIG. 3. (Color online) Violations for the generalized GHZ state: Here $B_{N,M} > 1$ indicates violation of the Bell inequality with (from lower to top line) $N = 3,4,5,6$ sites and $M = 2$ measurement settings for a detector efficiency of $\eta = 0.75\%$. The results for settings $M =$ 2–8 are indistinguishable.

We find that violations are possible over a significant range of $\alpha$, though the range is a more restricted range than that obtainable for the MABK inequalities. This is not unexpected, given that the inequalities here have a smaller violation ratio $B$ and do not violate LHV theories at all for $N = 2$. As with the MABK case, the range of $\alpha$ giving a violation expands with $N$, but in this case the results are essentially unchanged with $M$. Our numerical results suggest that the respective ranges of $\alpha$ coincide up to a shift of $N$ by one; that is, our numerical results are consistent with the relation $\sin 2\alpha \leqslant 1/\sqrt{2^{N-2}}$ for the SV inequalities.

## VI. DISCUSSION AND CONCLUSION

We have presented multisetting Bell inequalities that allow violation for an $N$-partite GHZ state when the efficiencies at each site approach $\eta \to 50\%$. This supplements the earlier work by Pal *et al.*, which reported multisetting Bell violations for reasonable efficiencies (as low as 38% for eight sites and 11 settings) [21]. We show that for the inequalities considered in this paper, an efficiency limit of arbitrarily close to 50% is attainable independently of the number of measurement settings and for the maximally entangled $N$-partite GHZ state that has a symmetric weighting of its composite spin states, provided one uses a sufficiently large number of sites $N$. The inequalities used here are different from those of Pal *et al.*, in that the "no detection" event is assigned the outcome of zero that is distinct from the Pauli spin values.

The violations are obtained using the spin version of the Shchukin-Vogel inequalities. These, however, require three or more sites ($N \geqslant 3$) in order to allow a test of Bell nonlocality even at ideal efficiencies. By comparison, our studies for this case suggest that violations are not possible at such low-efficiency thresholds for multipartite $W$ states. We also note that the natural multipartite extensions of Gisin's original multisetting inequalities [15] would not lead to lower threshold efficiencies.


## ACKNOWLEDGMENTS

We wish to acknowledge funding support from the Australian Research Council through the Discovery Grants program.



[1] J. S. Bell, Physics **1**, 195 (1964).
[2] J. F. Clauser and A. Shimony, Rep. Prog. Phys. **41**, 1881 (1978).
[3] S. J. Freedman and J. F. Clauser, Phys. Rev. Lett. **28**, 938 (1972).
[4] A. Aspect, J. Dalibard, and G. Roger, Phys. Rev. Lett. **49**, 1804 (1982).
[5] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, and A. Zeilinger, Phys Rev. Lett. **81**, 5039 (1998).
[6] G. Svetlichny, Phys. Rev. D **35**, 3066 (1987); D. Collins, N. Gisin, S. Popescu, D. Roberts, and V. Scarani, Phys. Rev. Lett. **88**, 170405 (2002).
[7] D. M. Greenberger, M. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Theory and Conceptions of the Universe*, edited by M. Kafatos (Kluwer Dordrecht, The Netherlands, 1989).
[8] N. D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990).
[9] S. L Braunstein and A. Mann, Phys Rev. A **47**, R2427 (1993); G. Brassard *et al.*, Quantum Inf. Comput. **5**, 538 (2005).
[10] C. Y. Lu *et al.*, Nat. Phys. **3**, 91 (2007); Y. F. Huang *et al.*, Nat. Commun. **2**, 546 (2011); X. C. Yao *et al.*, Nat. Photon. **6**, 225 (2012); H. Lu *et al.*, *ibid.* **8**, 364 (2014).
[11] J.-W. Pan, D. Bouwmeester, M. Daniell, H. Weinfurter, and A. Zeilinger, Nature (London) **403**, 515 (2000).
[12] J. Lavioe *et al.*, New J. Phys **11**, 073051 (2009); H. X. Lu, J. Q. Zhao, X. Q. Wang, and L. Z. Cao, Phys. Rev. A **84**, 012111 (2011); B. P. Lanyon, M. Zwerger, P. Jurcevic, C. Hempel, W. Dür, H. J. Briegel, R. Blatt, and C.F. Roos, Phys. Rev. Lett. **112**, 100403 (2014).
[13] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).
[14] N. Brunner *et al.*, Rev. Mod. Phys. **86**, 419 (2014).
[15] N. Gisin, Phys. Lett. A **260**, 1 (1999).
[16] D. Collins and N. Gisin, J. Phys. A: Math. Gen. **37**, 1775 (2004).
[17] A. Ekert, Phys. Rev. Lett. **67**, 661 (1991); V. Scarani and N. Gisin, *ibid.* **87**, 117901 (2001); J. Barrett, L. Hardy, and A. Kent, *ibid.* **95**, 010503 (2005); A. Acin, N. Brunner, N. Gisin, S. Massar, S. Pironio, and V. Scarani, *ibid.* **98**, 230501 (2007).
[18] P. Pearle, Phys. Rev. D **2**, 1418 (1970).
[19] M. Giustina *et al.*, Nature (London) **497**, 227 (2013); B. G. Christensen *et al.*, Phys. Rev. Lett. **111**, 130406 (2013).
[20] N. Brunner, N. Gisin, V. Scarani, and C. Simon, Phys. Rev. Lett. **98**, 220403 (2007).
[21] K. F. Pál, T. Vértesi, and N. Brunner, Phys. Rev. A **86**, 062111 (2012).
[22] J. F. Clauser and M. A. Horne, Phys. Rev. D **10**, 526 (1974).
[23] P. H. Eberhard, Phys. Rev. A **47**, R747 (1993).
[24] S. Massar and S. Pironio, Phys. Rev. A **68**, 062109 (2003).
[25] T. Vertesi, S. Pironio, and N. Brunner, Phys. Rev. Lett. **104**, 060401 (2010).
[26] M. D. Reid, Phys. Rev. A **87**, 062108 (2013).
[27] M. D. Reid, Phys. Rev. A **88**, 062108 (2013).
[28] A. Cabello and J. A. Larsson, Phys. Rev. Lett. **98**, 220402 (2007).
[29] H. M. Wiseman, S. J. Jones, and A. C. Doherty, Phys. Rev. Lett. **98**, 140402 (2007); S. J. Jones, H. M. Wiseman, and A. C Doherty, Phys. Rev. A **76**, 052116 (2007); E. G. Cavalcanti, S. J. Jones, H. M. Wiseman, and M. D. Reid, *ibid.* **80**, 032112 (2009).
[30] A. J. Bennet, D. A. Evans, D. J. Saunders, C. Branciard, E. G. Cavalcanti, H. M. Wiseman, and G. J. Pryde, Phys. Rev. X **2**, 031003 (2012).
[31] D. Smith *et al.*, Nat. Commun. **3**, 625 (2012); B. Wittmann *et al.*, New J. Phys. **14**, 053030 (2012).
[32] N. Gisin and B. Gisin, Phys. Lett. A **260**, 323 (1999).
[33] J. A. Larsson and J. Semitecolos, Phys Rev. A **63**, 022117 (2001); A. Cabello, D. Rodriguez, and I. Villanueva, Phys. Rev. Lett. **101**, 120402 (2008).
[34] E. G. Cavalcanti, Q. Y. He, M. D. Reid, and H. M. Wiseman, Phys. Rev. A **84**, 032115 (2011).
[35] E. Shchukin and W. Vogel, Phys. Rev. A **78**, 032104 (2008).
[36] E. G. Cavalcanti, C. J. Foster, M. D. Reid, and P. D. Drummond, Phys. Rev. Lett. **99**, 210405 (2007).
[37] A. Salles *et al.*, Quantum Inf. Comput. **10**, 0703 (2010).
[38] M. Ardehali, Phys. Rev. A **46**, 5375 (1992).
[39] A. V. Belinskii and D. N. Klyshko, Phys. Usp. **36**, 653 (1993); N. Gisin and H. Bechmann-Pasquinucci, Phys. Lett. A **246**, 1 (1998).
[40] M. Zukowski, C. Brukner, W. Laskowski, and M. M. Wiesniak, Phys. Rev. Lett. **88**, 210402 (2002).
[41] X- H. Wu and H- S. Zong, Phys. Rev. A **68**, 032102 (2003).
[42] W. Laskowski, T. Paterek, M. Zukowski, and C. Brukner, Phys. Rev. Lett. **93**, 200401 (2004).
[43] Q. Y. He, E. G. Cavalcanti, M. D. Reid, and P. D. Drummond, Phys Rev A **81**, 062106 (2010).
[44] Q. Y. He, E. G. Cavalcanti, M. D. Reid, and P. D. Drummond, Phys. Rev. Lett. **103**, 180402 (2009).
[45] K- P. Marzlin and T. A. Osborn, arXiv:1202.2534.
[46] Q. Y. He, P. D. Drummond, and M. D. Reid, Phys. Rev. A **83**, 032120 (2011).
[47] Q. Y. He, P. D. Drummond, M. K. Olsen, and M. D. Reid, Phys. Rev. A **86**, 023626 (2012); Q. Y. He, M. D. Reid, T. G. Vaughan, C. Gross, M. Oberthaler, and P. D. Drummond, Phys. Rev. Lett. **106**, 120405 (2011).
[48] M. D. Reid, W. J. Munro, and F. DeMartini, Phys. Rev. A **66**, 033801 (2002).
[49] L. Rosales-Zarate, B. Opanchuk, P. D. Drummond, and M. D. Reid, Phys. Rev. A **90**, 022109 (2014).
[50] P. Walther, M. Aspelmeyer, and A. Zeilinger, Phys. Rev. A **75**, 012313 (2007).
[51] A. Garg and N. D. Mermin, Phys. Rev. D **35**, 3831 (1987).